Assignment 5

1. Let $g(z)=(z-z_2)(z-z_3)\cdots(z-z_m)$ which is analytic in $\mathbb{C}$.

$$\int_c \frac{1}{f(z)}dz = \int_C \frac{1}{(z-z_1)g(z)}dz = 2\pi i \frac{1}{g(z_1)} = \frac{2\pi i}{(z_1-z_2)(z_1-z_3)\cdots(z_1-z_n)}.$$

2. (a) let $f(z)=\dfrac{z^2}{(z^2+9)(z^2+4)^2}$

$z=\pm 3i,\ \pm 2i$ are 1,2 order singular points

Consider C_R as the semi circle $|z|=R$ in the upper half plane, $\because R>6$. Then the points $2i$, $3i$ are bounded by C_R and $z=x\ (-R\le x\le R)$

By residue theorem

$$\int_{C_R} f(z)dz+\int_{-R}^{R} f(z)dz = 2\pi i\, Res(f,3i) + 2\pi i\, Res(f,2i)$$

$$Res(f,3i) = \left.\frac{z^2}{(z+3i)(z^2+4)^2}\right|_{z=3i}$$

$$= \frac{-9}{6i(-5)^2} = \frac{-9}{150i} = \frac{3i}{50}$$

$$Res(f,2i) = \left(\frac{z^2}{(z^2+9)(z+2i)^2}\right)'_{2i} = \frac{36iz-2z^4}{(z^2+9)^2(z+2i)^3}$$

$$= \frac{-72-2\times 16}{25\times 4i\times(-16)} = \frac{-9-4}{100i\times(-2)} = \frac{13}{200i}$$

$$\therefore \int_{C_R} f(z)dz+\int_{-R}^{R} f(z)dz = \frac{-6\pi}{50}+\frac{13\pi}{100} = \frac{\pi}{100}$$

$$\left|\int_{C_R} f(z)dz\right| = \left|\int_{C_R} \frac{z^2}{(z^2+9)(z^2+4)^2}dz\right| \le \int_{C_R} \frac{|z|^2}{|z^2+9||z^2+4|^2}dz$$

$$\le \int_{C_R} \frac{R^2}{(R^2-9)(R^2-4)^2}dz = \frac{\pi R^3}{(R^2-9)(R^2-4)^2} \xrightarrow{R\to+\infty} 0$$

$$\therefore \lim_{R\to+\infty} \int_{-R}^{R} f(z)dz = \frac{\pi}{100}$$

Meanwhile $f(z)$ is an even function $\therefore \lim_{R\to+\infty} \int_0^R f(z)dz = \frac{\pi}{200}$

$$\therefore \int_0^{+\infty} \frac{x^2}{(x^2+9)(x^2+4)^2}dx = \frac{\pi}{200}$$

(b) Let $f(z) = \dfrac{e^{iz}}{z^2+4z+5}$. $-2+i$ and $-2-i$ are two 1 order singular points of $f(z)$

Consider C_R as the semi circle $|z|=R$ in the upper half plane, $R>6$. Then the point $-2+i$ is bounded by C_R and $z=x$ $(-R\le x\le R)$

By Residue theorem

$$\int_{C_R} f(z)dz + \int_{-R}^{R} f(z)dz = 2\pi i \,\mathrm{Res}(f, -2+i)$$

$$\mathrm{Res}(f, -2+i) = \left.\frac{e^{iz}}{(z+2+i)}\right|_{-2+i} = \frac{e^{-2i-1}}{2i} = \frac{1}{e}\,\frac{\cos 2 - \sin 2\cdot i}{2i}$$

$$= \frac{1}{e}\,\frac{-\cos 2\cdot i - \sin 2}{2}$$

$$\therefore 2\pi i \,\mathrm{Res}(f, -2+i) = \frac{\pi}{e}(\cos 2 - i\sin 2)$$

Because z belongs to half upper plane. So $|e^{iz}| = |e^{i(a+bi)}| = |e^{-b}\cdot e^{ia}| \le 1$ $(\because b>0)$

$$|z^2+4z+5| = |(z+2-i)|\cdot|(z+2+i)|$$
$$\geq (|z|-|2-i|)(|z|-|2+i|)$$
$$= (|z|-\sqrt{5})^2$$

$$\therefore \left|\int_{C_R} f(z)dz\right| \leq \int_{C_R} \frac{1}{(R-\sqrt{5})^2}dz = \frac{\pi R}{(R-\sqrt{5})^2} \xrightarrow{R\to+\infty} 0$$

$$\therefore \lim_{R\to+\infty} \int_{-R}^{R} f(z)dz = \frac{\pi}{e}(\cos 2 - i\cdot\sin 2)$$

Comparing the imaginary parts on both sides

$$P.V. \int_{-\infty}^{+\infty} \frac{\sin x}{x^2+4x+5} = -\frac{\pi}{e}\cdot\sin 2.$$

(c) $\displaystyle\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} \overset{\theta=2\pi-\alpha}{=\!=\!=} \int_{\pi}^{2\pi} \frac{d\alpha}{(a+\cos\alpha)^2}$ (change of variables)

$$\therefore \int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2}$$

Consider $C: z=e^{i\theta}$ as the Unit circle centered at origin and positively oriented

$$dz = e^{i\theta}\cdot i d\theta = iz d\theta$$

$$\frac{1}{2}\int_0^{2\pi}\frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\int_C \frac{\frac{dz}{iz}}{\left(a+\frac{z+z^{-1}}{2}\right)^2} = \frac{1}{2}\int_C \frac{\frac{dz}{iz}}{\left(\frac{z^2+2az+1}{2z}\right)^2}$$

$$= \frac{2}{i}\int_C \frac{zdz}{(z^2+2az+1)^2} = \frac{2}{i}\int_C \frac{zdz}{(z+a-\sqrt{a^2-1})^2(z+a+\sqrt{a^2-1})^2}$$

$$\text{let } f(z) = \frac{z}{(z+a-\sqrt{a^2-1})^2(z+a+\sqrt{a^2-1})^2}$$

$\sqrt{a^2-1}-a$ is a singular point of f or order 2 lie in C

$$\text{So } \operatorname{Res}(f, \sqrt{a^2-1}-a) = \left(\frac{z}{(z+a+\sqrt{a^2-1})^2}\right)'\Bigg|_{z=\sqrt{a^2-1}-a}$$

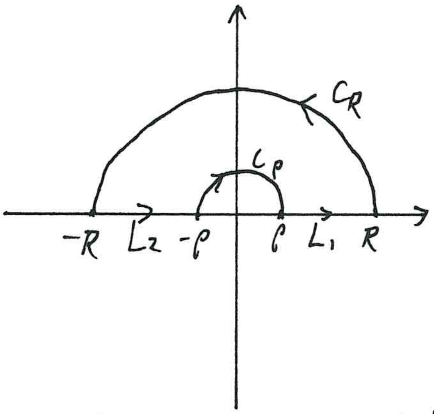
$$= \frac{(z+a+\sqrt{a^2-1})^2 - 2(z+a+\sqrt{a^2-1})\cdot z}{(z+a+\sqrt{a^2-1})^4}\Bigg|_{z=\sqrt{a^2-1}-a}$$

$$= \frac{-z+a+\sqrt{a^2-1}}{(z+a+\sqrt{a^2-1})^3}\Bigg|_{z=\sqrt{a^2-1}-a} = \frac{2a}{8(\sqrt{a^2-1})^3}$$

$$\therefore \int_C f(z)dz = \frac{4\pi i a}{8(\sqrt{a^2-1})^3}$$

$$\therefore \frac{2}{i}\int_C f(z)dz = \frac{\pi a}{(\sqrt{a^2-1})^3}$$

(d) Let $f(z) = \dfrac{z^a}{(z^2+1)^2} = \dfrac{e^{a\ln z}}{(z^2+1)^2}$ and $0 < \arg z < 2\pi$



$C = C_R \cup L_2 \cup C_\rho \cup L_1$

$$\int_C f(z)dz = 2\pi i \operatorname{Res}(f, i) = 2\pi i \left(\frac{z^a}{(z+i)^2}\right)'\Bigg|_{z=i}$$

$$= 2\pi i \, \frac{z^{a-1}[az+ai-2z]}{(z+i)^3}\Bigg|_{z=i} = 2\pi i \, \frac{e^{(a-1)\ln z}[az+ai-2z]}{(z+i)^3}$$

$$= 2\pi i \, \frac{e^{(a-1)\ln i}(2ai-2i)}{(2i)^3} = 2\pi i \, \frac{e^{(a-1)\frac{\pi i}{2}}(a-1)}{-4} = \pi \frac{e^{\frac{a\pi i}{2}}(a-1)}{-2}$$

$$\left| \int_{C_R} \frac{z^a}{(z^2+1)^2} dz \right| \le \int_{C_R} \frac{|z|^a}{|z^2+1|^2} dz \le \int_{C_R} \frac{R^a}{(R^2-1)^2} dz$$

$$= \frac{\pi R^{a+1}}{(R^2-1)^2} \xrightarrow[(\because a<3)]{R\to+\infty} 0$$

$$\left| \int_{C_\rho} \frac{z^a}{(z^2+1)^2} dz \right| \le \int_{C_\rho} \frac{|z|^a}{|z^2+1|^2} dz \le \frac{|\rho|^a \cdot \pi\rho}{(1-\rho^2)^2} \xrightarrow[(\because a>-1)]{\rho\to 0^+} 0$$

Meanwhile $\displaystyle\int_{-R}^{-\rho} \frac{x^a}{(x^2+1)^2} dx = \int_{-R}^{-\rho} \frac{e^{(\ln|-x|+i\pi)a}}{(x^2+1)^2} dx$

$$= e^{i\pi a} \int_{-R}^{-\rho} \frac{e^{\ln|-x|\cdot a}}{(x^2+1)^2} dx = e^{i\pi a} \int_{\rho}^{R} \frac{e^{\ln x\cdot a}}{(x^2+1)^2} dx$$

$$= e^{i\pi a} \int_\rho^R \frac{x^a}{(x^2+1)^2} dx$$

Hence $\displaystyle\int_C f(z)dz = \int_{C_R} f(z)dz + \int_{C_\rho} f(z)dz + \int_{-R}^{-\rho} \frac{x^a}{(x^2+1)^2} dx + \int_\rho^R \frac{x^a}{(x^2+1)^2} dx$

$$= \int_{C_R} f(z)dz + \int_{C_\rho} f(z)dz + (1+e^{i\pi a}) \int_\rho^R \frac{x^a}{(x^2+1)^2} dx$$

Let $R\to+\infty$ $\rho\to 0^+$, we have

$$\pi \frac{e^{\frac{a\pi i}{2}}(a-1)}{-2} = (1+e^{i\pi a}) \int_0^{+\infty} \frac{x^a}{(x^2+1)^2} dx$$

So $\displaystyle\int_0^{+\infty} \frac{x^a}{(x^2+1)^2} dx = \frac{\frac{\pi}{2}(1-a)e^{\frac{a\pi i}{2}}}{1+e^{i\pi a}} = \frac{\frac{\pi}{2}(1-a)}{e^{-\frac{a\pi i}{2}}+e^{\frac{a\pi i}{2}}} = \frac{\pi(1-a)}{4\cos\frac{a\pi}{2}}$

3. $f(z) = (z-z_1)^{m_1}(z-z_2)^{m_2}\cdots(z-z_n)^{m_n}g(z)$, where $g(z)$ has no zeros in $\mathbb{C}$

$$f'(z) = m_1(z-z_1)^{m_1-1}(z-z_2)^{m_2}\cdots(z-z_n)^{m_n}g(z) +$$
$$(z-z_1)^{m_1}m_2(z-z_2)^{m_2-1}\cdots(z-z_n)^{m_n}g(z) +$$
$$\vdots$$
$$(z-z_1)^{m_1}\cdots\cdots\cdot m_n(z-z_n)^{m_n-1}g(z) +$$
$$(z-z_1)^{m_1}(z-z_2)^{m_2}\cdots(z-z_n)^{m_n}g'(z)$$

$$\therefore z\frac{f'(z)}{f(z)} = \frac{m_1 z}{z-z_1} + \frac{m_2 z}{z-z_2} + \cdots + \frac{m_n z}{z-z_n} + \frac{z g'(z)}{g(z)}$$

$$\int_C \frac{zf'(z)}{f(z)}dz = \int_C \sum_{i=1}^{n}\frac{m_i z}{z-z_i}dz + \int_C \frac{zg'(z)}{g(z)}$$

$$= \int_C \sum_{i=1}^{n}\frac{m_i z}{z-z_i}dz + 0$$

$$\overset{\text{Residue theo}}{=} \sum_{i=1}^{n} 2\pi i\cdot m_i z_i = 2\pi i\sum_{k=1}^{n} m_k z_k$$

4 let Γ: circle centered at origin point with radius 2

$f(z) = z^5$

$g(z) = z^5+3z^3+7$

On Γ $|f(z)-g(z)| = |3z^3+7| \le 31 < 32 = |z^5|$

By Rouche theorem, $f(z)$ and $g(z)$ have same amount of zeros in Γ.

$f(z)$ has 5 zeros in $\Gamma \Rightarrow g(z)$ has 5 zeros in Γ

But the order of $g(z)$ is 5, $\Rightarrow$ all zeros of $g(z)$ are in the disk enclosed by Γ.